VELOCITY FIELD INDUCED IN A LIQUID BY A ROTATING CONE

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Research was carried out under the Bureau of Ships Fundamental Hydrodynamics Research Program, S-R009 01 01, administered by the David Taylor Model Basin.

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ABSTRACT

The present paper presents an analytical study of the steady-state velocity field induced in a liquid by a rotating cone. As a consequence of the experimental observations, velocity functions which very accurately define the flow have been developed. The functions are of such a nature that the centerline of a vortex and the flow about the centerline are accurately described. The boundary conditions at the surface of the rotating cone and at the fixed boundaries are satisfied. Also the continuity equation is satisfied.

Finally, the stress field in the fluid has been studied in terms of the velocity functions. In particular, the relation between angular velocity of cone and the total torque on the cone was determined by precise experiments. The relation is linear and a discussion of the implications of this fact in terms of the nature of the constitutive equations for the liquid is presented. The force of the evidence is that the liquids studied are Newtonian and the viscosity coefficients determined by experiment agree with those cited by various authorities.

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I. INTRODUCTION

In 1953, there was introduced an experimental apparatus which has become known as the Ferranti-cone-plate viscometer[1]. Determination of coefficients of viscosity of liquids tested with this apparatus is based upon the assumption that a liquid always flows in concentric circular paths about the axis of rotation of the cone. In October 1962, in a technical paper entitled, "A Rotational Fluid Flow Generator for Studies in Rheology"[2], it was shown that actually a more complex flow pattern exists for all speeds investigated. It was further shown that the observed flow is stable at all velocities in the range studied, even down to zero velocity. Hence, in particular it is not of the Taylor instability type which is generated between two rotating coaxial cylinders.

Motivated by these discoveries, it was decided to study the actual flow pattern and to make a comprehensive analysis of the corresponding steady-state velocity field. An effective method available for the study is based upon direct observation

of the flow in the above mentioned generator. Flow visualization techniques and torque measurements reveal a considerable amount of information concerning the nature of the flow. Using the knowledge so gained, functions representing the velocity components were constructed.

II. THE VELOCITY FIELD

The structure of the velocity field was graphically shown by visualization techniques, as in Figure 1. It is this picture which clearly demonstrates the vortical nature of the flow in considerable detail. Starting with such pictures one can develop expressions for the components of velocity, analytically defining the flow.

In a cross-section, the velocity trajectories are a family of closed curves. Using cylindrical coordinates, a conceivable representation of these trajectories for a ϕ = constant plane is:

$$k = \sum_{m=0}^{m=m_1} \sum_{n=0}^{n=n_1} a_{mn} r^m z^n = \text{constant} \qquad (1)$$

In non-dimensionalized coordinates a particular form of the series is:

$$k = -(Z-R)^{\lambda_1} Z^{\lambda_2} (1-R)^{\lambda_3} + (Z_c - R_c)^{\lambda_1} Z_c^{\lambda_2} (1-R_c)^{\lambda_3} \qquad (2)$$

which is suggested by the equation of the boundary,

$$(Z-R)Z(1-R) = 0 \tag{3}$$

where the non-dimensionalized coordinates and velocities are:

$$R = \frac{r}{r_0}, \quad \phi = \phi, \quad Z = \frac{z}{hr_0}$$
$$u_R = \frac{v_r}{r_0\Omega_0}, \quad u_\phi = \frac{v_\phi}{r_0\Omega_0}, \quad u_Z = \frac{v_z}{hr_0\Omega_0} \tag{4}$$

The maximum radius of the cone is r_0 and the angular velocity of the cone is Ω_0 . The tangent of the angle between the surface of the cone and the horizontal surface of the cylindrical container is h . In the future this angle will be referred to as the cone angle and will be denoted by the letter α . Also (R_c , Z_c) are the coordinates of the vortex center.

For these trajectories the slope of any tangent line is the ratio of the velocity components at the point of tangency. Also the velocity components should be such as to satisfy the continuity equation. Hence, the velocity components u_R and u_Z defined in terms of k are:

$$u_R = \frac{E}{R}\frac{\partial k}{\partial Z} \qquad u_Z = -\frac{E}{R}\frac{\partial k}{\partial R} \tag{5}$$

where E is a function of k , and k is a function of R and Z , given by equation (2). Thus,

$$u_R = -\frac{E}{R}(1-R)^{\lambda_3} Z^{\lambda_2 - 1}(Z-R)^{\lambda_1 - 1}[(\lambda_1 + \lambda_2)Z - \lambda_2 R] \tag{6}$$

and

$$u_Z = -\frac{E}{R}(1-R)^{\lambda_3 - 1} Z^{\lambda_2}(Z-R)^{\lambda_1 - 1}[\lambda_3 Z - (\lambda_1 + \lambda_3)R + \lambda_1] \tag{7}$$

Furthermore, observation shows that at the vortex center, u_ϕ is not zero, but u_R and u_Z are zero. Hence, from the expressions for u_R and u_Z ,

$$\begin{aligned} &(\lambda_1 + \lambda_2)Z_c - \lambda_2 R_c = 0 \\ &\text{and} \\ &\lambda_3 Z_c - (\lambda_1 + \lambda_3)R_c + \lambda_1 = 0 \end{aligned} \tag{8}$$

Consequently, the following conditions are placed on λ_1 , λ_2 , λ_3 :

$$\frac{\lambda_1}{\lambda_2} = \frac{R_c - Z_c}{Z_c} \tag{9}$$

$$\frac{\lambda_1}{\lambda_3} = \frac{Z_c - R_c}{R_c - 1} \tag{10}$$

Furthermore, because u_R and u_Z are zero on the boundary the λ's must be chosen so that,

$$\lambda_1 - 1 > 0 \quad , \qquad \lambda_2 - 1 > 0 \quad , \qquad \lambda_3 - 1 > 0 \tag{11}$$

It follows that u_R will be zero at every point in the flow domain where:

$$Z = \frac{Z_c}{R_c} R \tag{12}$$

and u_Z will be zero at every point where:

$$Z = \frac{Z_c - 1}{R_c - 1} R - \frac{Z_c - R_c}{R_c - 1} \tag{13}$$

For values of $Z > \frac{Z_c}{R_c} R$, u_R is positive and for values of $Z < \frac{Z_c}{R_c} R$, u_R is negative. Likewise, for values of $Z > [\frac{Z_c - 1}{R_c - 1} R - \frac{Z_c - R_c}{R_c - 1}]$, u_Z is positive and for values of $Z < [\frac{Z_c - 1}{R_c - 1} R - \frac{Z_c - R_c}{R_c - 1}]$, u_Z is negative. These results agree with the observations using flow visualization techniques.

The third velocity component u_ϕ could conceivably be represented as follows:

$$u_\phi = \frac{Z(1-R)}{1 - Z} + \sum_{m=0}^{m=m_1} \sum_{n=0}^{n=n_1} b_{mn} R^m Z^n \tag{14}$$

A particular simple form of this series was chosen to represent u_ϕ as follows:

$$u_\phi = \frac{Z(1-R)}{1-Z} + AZ(Z-R)(1-R)(R+BZ) \qquad (15)$$

where A and B are constants to be determined. Now u_R, u_ϕ, u_Z have been determined in such a manner that the continuity equation and the boundary conditions are satisfied. The constants λ_1, λ_2, λ_3, A, B and the parameter E can now be determined by experiment.

III. METHODS FOR THE DETERMINATION OF THE λ's, E, A, AND B

A. The Determination of the λ's

The λ's can be determined once the position of the vortex center is known. The following is a description of the experimental method used to obtain this information.

Small water particles were injected into the flow domain with a hypodermic needle. Using two micrometer screws, the coordinates of the particle with reference to the top and side of the cylindrical container could be determined. Water particles moving in a circular path about the axis of rotation of the cone are at the vortex center. For castor oil U.S.P. the results are presented in Table I. The distance of the vortex center from the vertical wall is ξ, and the distance of the vortex center from the top of the cylindrical container is η. The experimental arrangement for the cylindrical container and

one of the cones is shown in Figure 2.

From the table it can be seen that as the cone angle decreases, the vortex center moves toward the cylindrical wall, It is concluded that as α , the cone angle, approaches zero degrees, R_c approaches unity. As α increases, the vortex center moves toward the axis of rotation of the cone. Some observations made with a 76° angle cone indicate that such is the case also for cones whose angles are larger than 45°.

B. The Determination of E

The quantity E varies with the surface parameter k and the angular velocity of the cone. Its determination was made from considerations of what may be called types of circulation of a particle of the liquid. Specifically, the circulations considered are of two kinds; one around a cross-sectional trajectory, the other around the axis of rotation of the cone. This follows from the fact that a given particle of liquid always moves on a particular toroidal surface. In so doing the particle moves not only around the fixed circular axis which coincides with the vortex center in the liquid but also around the axis of rotation of the cone.

The time required for a particle to make one revolution about the fixed circular axis is:

$$t = \oint \frac{ds}{v} \qquad (15)$$

where ds is a differential length on a particular trajectory defined by k, and v is the magnitude of the vector sum of the velocity components v_r and v_z. Also, the above time can be related to the average angular velocity, $\omega_{\alpha i}$, of the particle as it rotates once about the fixed circular axis. The symbol α_i refers to a particular surface defined by the parameter k. The relation is:

$$\frac{2\pi}{\omega_{\alpha i}} = \oint \frac{ds}{v} \qquad (17)$$

Non-dimensionalizing the integrand and solving for E one finds the relation:

$$E = \frac{\omega_{\alpha i}}{2\pi\Omega_0} \oint \frac{[dR^2 + h^2\ dZ^2]^{1/2}}{[\bar{u}_R{}^2 + h^2\ \bar{u}_Z{}^2]^{1/2}} \qquad (18)$$

where $\bar{u}_R = \frac{u_R}{E}$, $\bar{u}_Z = \frac{u_Z}{E}$.

It was found experimentally that the average angular velocity of the particle about the cone axis, $\Omega_{\alpha i}$, is related to the average angular velocity $\omega_{\alpha i}$ of the particle about the fixed circular axis as follows:

$$\frac{\omega_{\alpha i}}{\Omega_{\alpha i}} = R_{\alpha i} \cdot \frac{\Omega_0}{\Omega_s} \qquad (19)$$

where $R_{\alpha i}$ is a parameter depending only upon k. The maximum angular velocity of the cone for the velocity range

studied is Ω_s . Also it was experimentally found that $\Omega_{\alpha i}$ is related to the angular velocity of the cone as follows:

$$\Omega_{\alpha i} = G\,\Omega_0 \tag{20}$$

where G is a parameter depending only upon k . Using equations (19) and (20) with equation (18) the parameter E is given as follows:

$$E = \frac{R_{\alpha i}(k)G(k)}{2\pi}\ \frac{\Omega_0}{\Omega_s} \oint \frac{[dR^2 + h^2\ dZ^2]^{1/2}}{[\bar{u}_R{}^2 + h^2\ \bar{u}_Z{}^2]^{1/2}} \tag{21}$$

The two parameters $R_{\alpha i}$ and G can be determined experimentally by measuring the time required for the particle to travel once about a trajectory defined by k , and by counting the number of times the particle moves about the cone axis while moving once about the plane trajectory. The integral can be evaluated by using the derived velocity functions.

C. The Determination of A and B

The constants A and B can be calculated using data from two experimental measurements. The first measurement is of the angular velocity of water particles placed at the vortex center and the second measurement is the torque on the cone surface.

Using a stop watch over long periods of observation, the angular velocity of water particles placed at the vortex center was determined. The results are shown in Table I. The ratio

of the angular velocity of the particle at the vortex center to the angular velocity of the cone is Ω_c'. The values in Table I are the averages of all the readings taken and $\Delta\Omega_c'$ is the maximum percent of variation between the average and an individual reading. The ratio Ω_c' is a constant over a wide range of angular velocity. The range for which the results are listed in Table I was between 10 rpm and 80 rpm. The difference in individual readings is probably caused by the fact that the water particle cannot be placed at the exact center of the vortex. It was found that by decreasing the size of the water particle as the angular velocity of the cone decreases, the agreement between individual readings was improved.

As the cone angle approaches zero degrees, the angular velocity at the vortex center approaches one-half the value of the angular velocity of the cone. As the angle increases, the ratio Ω_c' decreases. For $\alpha = 45^{\circ}$, the ratio is 0.306.

The torque on the rotating shaft was measured by a transducer constructed and developed for the R.P.I. Rotational Fluid Flow Generator. The description of this transducer and the co-related electrical circuit is completely described in the previously mentioned technical report[2]. The torque was measured for several cone angles over the range of angular velocity from zero to 100 rpm. The liquids investigated were castor oil U.S.P., S.A.E. 10W, S.A.E. 20-20W, and S.A.E. 30 Lubrite Motor Oil manufactured by the Mobile Oil Company. Using the simplest

means of thermal shielding and insulation, the experiments were carried out at a temperature of 22.2° Centigrade. The results of these experiments show that the torque is a linear function of the angular velocity of the cone. Hence, the constitutive equation for a Newtonian, incompressible liquid was considered to be the constitutive equation characterizing the liquids for which the developed velocity field is applicable. The equation is written:

$$T = - pI + 2\mu X \tag{22}$$

where T and X are the stress tensor and the rate of deformation tensor respectively, I is a unit tensor, and p is defined as the arithmetical mean of the normal stresses. The coefficient of viscosity is μ.

In cylindrical coordinates, the shear stress component which produces a torque on the surface of the cone is as follows:

$$T_{\theta\phi} = \mu\left[\left(\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right)\sin\beta - \frac{\partial v_\phi}{\partial z}\sin\beta \right] \tag{23}$$

The total torque on the surface of the cone is then:

$$M = 2\pi \int_0^a r^2 \left[T_{\theta\phi} \right]_{z=r\tan\alpha} \cdot \sec\alpha \, dr \tag{24}$$

Putting the velocity functions into equation (24) the following formula was derived.

$$\frac{M/\Omega_0 \cdot \cos\alpha \sin\alpha}{2\pi\mu r_0^3} = \left\{\frac{(1-R)^2}{2} - 2(1-R) + \log(1-R) - \frac{\delta R^5}{5} + \frac{\delta R^6}{6}\right\}_0^a \tag{25}$$

where:

$$\delta = A(1+B) \tag{26}$$

For the flow examined experimentally, the length of the free surface was kept at 10% of the total length of the largest radius of the cone. Hence in equation (25) the limit a is taken to be 0.9. Then equation (25) becomes:

$$\frac{M/\Omega_0 \cdot \cos\alpha \sin\alpha}{2\pi\mu r_0^3} = -(.998 + .0306) \tag{27}$$

Using the results of the torque experiments, the expression on the left hand side of equation (27) was evaluated. The results are shown in Table II. Using the average value for each column, δ can be calculated for each angle using equation (27). The results are given in Table III.

Using the values of δ in Table III, the values for the coordinates of the vortex center, and the values of the angular velocity of particles placed at the vortex center, given in Table I, the values of A and B can be determined as follows:

$$\left[\frac{u_\phi}{R}\right]_{\substack{R = R_c \\ Z = Z_c}} = \Omega_c' = \frac{Z_c(1-R_c)}{R_c(1-Z_c)} + \frac{AZ_c}{R_c}(Z_c-R_c)(1-R_c)(R_c+BZ_c) \tag{28}$$

and,

$$A = \frac{\delta}{1 + B} \tag{29}$$

Solving equations (28) and (29) simultaneously A and B are given by the following equations:

$$A = -\frac{\Omega_c' R_c(1-Z_c) - Z_c(1-R_c) - \delta Z_c^2 R_c(Z_c-R_c)(1-R_c)(1-Z_c)}{Z_c R_c(1-R_c)(1-Z_c)(Z_c-R_c)^2} \tag{30}$$

$$B = -\frac{\Omega_c' R_c(1-Z_c) - Z_c(1-R_c) - \delta Z_c R_c^2(Z_c-R_c)(1-R_c)(1-Z_c)}{\Omega_c' R_c(1-Z_c) - Z_c(1-R_c) - \delta Z_c^2 R_c(Z_c-R_c)(1-R_c)(1-Z_c)} \tag{31}$$

IV. ILLUSTRATION OF THE METHOD FOR DETERMINATION OF THE VARIOUS PARAMETERS

Using the experimental data of Table I, and the equations (9) and (10) for $\alpha = 35^\circ$, the λ's are as follows:

$$\lambda_1 = \lambda_2 = 4 \qquad \lambda_3 = 2 \tag{32}$$

The equation of the velocity trajectories is:

$$k = -(Z-R)^4 Z^4 (1-R)^2 + 2.62 \times 10^{-5} \tag{33}$$

In Figure 3, several curves have been plotted using fractional parts of 2.62×10^{-5} for k. A comparison can be made

between the calculated trajectories shown in Figure 3, and the actual trajectories shown in Figure 1.

Using the experimental data given in Figures 4, 5, 6, and the curves drawn in Figure 3, the parameter E was determined. For curve "a", $E = 13.6\Omega_0$, for curve "c", $E = 8.65\Omega_0$, and for curve "d", $E = 1.25\Omega_0$.

Using the experimental data of Tables I, II, and III, for α equal 35° it was found that:

$\delta = 5.00$

$\Omega_c' = .35$

$R_c = .8$ and $Z_c = .4$.

Using equations (30) and (31) it was found that:

$A = -19.6$ and $B = -1.25$.

Hence, the velocity components are:

$$u_R = -\frac{4E}{R}(1-R)^2 Z^3 (Z-R)^3 (2Z-R) \tag{34}$$

$$u_Z = -\frac{2E}{R}(1-R) Z^4 (Z-R)^3 (Z-3R+2) \tag{35}$$

$$v_\phi = \frac{Z(1-R)}{1-Z} - 19.6Z(Z-R)(1-R)(R-1.25Z) \tag{36}$$

A formula for finding the coefficient of viscosity of liquids using a 35° angle cone was developed using the results of Table II. It is as follows:

$$\mu = .0256 \frac{M}{\Omega_0} \text{ centipoise} \tag{37}$$

where M is in dyne-cm and Ω_0 is in $\sec^{-1}$. The coefficients of viscosity calculated from the formula are given in Table IV. The value of the coefficient of viscosity for castor oil U.S.P. compares well with that given in the Smithsonian Tables[3] and the viscosities of the S.A.E. oils agree with the values given by Shaw[4].

Also it was found that for a 35° angle cone the circulation time about the fixed axis through the vortex center decreased as the vortex center was approached. For Ω_0 equal to 40 rpm the circulation time for curve "a" was 1.57 minutes, and for curve "d" the value was .39 minutes. An estimate was made of the time required for a particle to travel about a curve from $R = .3$ to $R = .9$ for a cone whose angle was 4°. For Ω_0 equal to 40 rpm the circulation time was of the order of 10^{26} minutes. Hence as the angle of the cone decreases to zero it is concluded that the circulation approaches zero.

V. EQUATIONS OF MOTION

Applying the constitutive equation (22) to Newton's equations of motion for axisymmetric, steady-state flow the equations of motion which have the linear Stokesian hypothesis for a constitutive equation are the Navier-Stokes equations. In cylindrical coordinates, the non-dimensionalized equations for the specific problem under consideration, neglecting body forces are:

$$u_R \frac{\partial u_R}{\partial R} - \frac{u_\phi^{\ 2}}{R} + u_Z \frac{\partial u_R}{\partial Z} = \frac{1}{K}\left\{- \frac{\partial \overline{P}}{\partial R} + \frac{\partial^2 u_R}{\partial Z^2} + h^2\left[\frac{\partial^2 u_R}{\partial R^2} + \frac{1}{R}\frac{\partial u_R}{\partial R} - \frac{u_R}{R^2}\right]\right\} \tag{38}$$

$$u_R \frac{\partial u_\phi}{\partial R} + \frac{u_R u_\phi}{R} + u_Z \frac{\partial u_\phi}{\partial Z} = \frac{1}{K}\left\{h^2\left[\frac{\partial^2 u_\phi}{\partial R^2} + \frac{1}{R}\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R^2}\right] + \frac{\partial^2 u_\phi}{\partial Z^2}\right\} \tag{39}$$

$$u_R \frac{\partial u_Z}{\partial R} + u_Z \frac{\partial u_Z}{\partial Z} = - \frac{1}{Kh^2}\frac{\partial \overline{P}}{\partial Z} + \frac{1}{K}\left\{h^2\left[\frac{\partial^2 u_Z}{\partial R^2} + \frac{1}{R}\frac{\partial u_Z}{\partial R}\right] + \frac{\partial^2 u_Z}{\partial Z^2}\right\} \tag{40}$$

where $\overline{P}$ is related to the arithmetical mean p of the normal stresses by the relation

$$p(r,z) = \frac{\mu\Omega_0\overline{P}}{h^2} \tag{41}$$

and K is defined as,

$$K = \frac{\Omega_0 r_0^{\,2} h^2}{\nu} \quad . \tag{42}$$

The Reynolds' number Re is proportional to K and given as follows;

$$Re = \frac{K}{h^2} \quad . \tag{43}$$

The Reynolds' number refers only to a cone of given angle and given gap width between the cone and cylindrical wall at the free surface. For a 35° angled cone, the corresponding range of the Reynolds' number using castor oil is

$$.48 \leq Re \leq 48 \tag{44}$$

The theoretical velocity field closely approximates the observed velocity trajectories in a ϕ = constant plane, and satisfies the continuity equation and the boundary conditions. Also, the derived velocity components indicate that the particles of the fluid spiral around the central curve of a vortex as they move about the axis of the cone. Also, the theoretical velocity components agree with the experimentally determined values at the vortex center. Using the velocity component in the ϕ direction, the total torque calculated on the cone is approximately the same as that calculated on the cylindrical container. It is on the basis of the agreement of these results with the actual flow produced in the laboratory that the

theoretically determined velocity field is considered to be an approximate solution of the differential equations (38), (39) and (40).

Furthermore, integrating equation (40) with respect to Z and equation (38) with respect to R, two representations of the non-dimensional pressure field are obtained.

$$\overline{P}_1 = \int \left\{ \frac{\partial^2 u_R}{\partial Z^2} + h^2 \left[\frac{\partial^2 u_R}{\partial R^2} + \frac{1}{R} \frac{\partial u_R}{\partial R} - \frac{u_R}{R^2} \right] - K \left[u_R \frac{\partial u_R}{\partial R} - \frac{{u_\phi}^2}{R} + u_Z \frac{\partial u_R}{\partial Z} \right] \right\} dR + F_1(Z) \tag{45}$$

$$\overline{P}_2 = \int \left\{ h^4 \left[\frac{\partial^2 u_Z}{\partial R^2} + \frac{1}{R} \frac{\partial u_Z}{\partial R} \right] + h^2 \left[\frac{\partial^2 u_Z}{\partial Z^2} \right] - Kh^2 \left[u_R \frac{\partial u_Z}{\partial R} + u_Z \frac{\partial u_Z}{\partial Z} \right] \right\} dZ + F_2(R) \tag{46}$$

If the velocity components u_ϕ, u_R, u_Z are exact solutions of the Navier-Stokes equations then $\overline{P}_1$ should, of course, equal $\overline{P}_2$.

The discrepancy that may exist between $\overline{P}_1$ and $\overline{P}_2$ in the present investigation has not been examined. It is an object of study for future research.

Finally it is suggested that an infinite series expansion of k and u_ϕ of the form:

$$k = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} R^m Z^n \tag{47}$$

and

$$u_\phi = \frac{Z(1-R)}{1-Z} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D_{mn} R^m Z^n \tag{48}$$

may provide an exact solution to the equations of motion.

VI. DISCUSSION AND CONCLUSION

Using observations of the actual flow patterns and measurements of torque and angular velocity, it was possible in the present study to determine satisfactorily the functions describing the velocity field induced in a liquid by a rotating cone. The stress field and the constitutive equations were studied in terms of the velocity functions. The relation between the angular velocity of the cone and the total torque was shown to be linear. Therefore, it was concluded that the liquids used in the investigation were Newtonian and the differential equations of motion governing the flow are the classical Navier-Stokes equations. Because of the many agreements of the constructed velocity functions with the actual flow for cones of any angle,

they are considered to be reasonable approximations to those which are a solution to the Navier-Stokes equations. As the angle of the cone approaches zero as a limit, the circulation about the vortex center was shown to approach zero. The torque and angular velocity relation approaches that which has been developed by rheologists for the determination of the coefficient of viscosity for Newtonian liquids, when rotating the liquids by cones whose angles are less than 4° . However, by using the velocity functions developed in the present research, the measurement of viscosity is now not limited to the use of cones with small angles and small amounts of liquid but can be carried out with cones of any angle rotating in any amount of liquid.

It is finally concluded that the investigation provides effective new knowledge of the flow generated by a rotating cone immersed in a liquid and furthermore, provides the rational basis for the generalization of formulae now used in viscosimetry for Newtonian liquids.

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Figure 1. Structure of the Flow

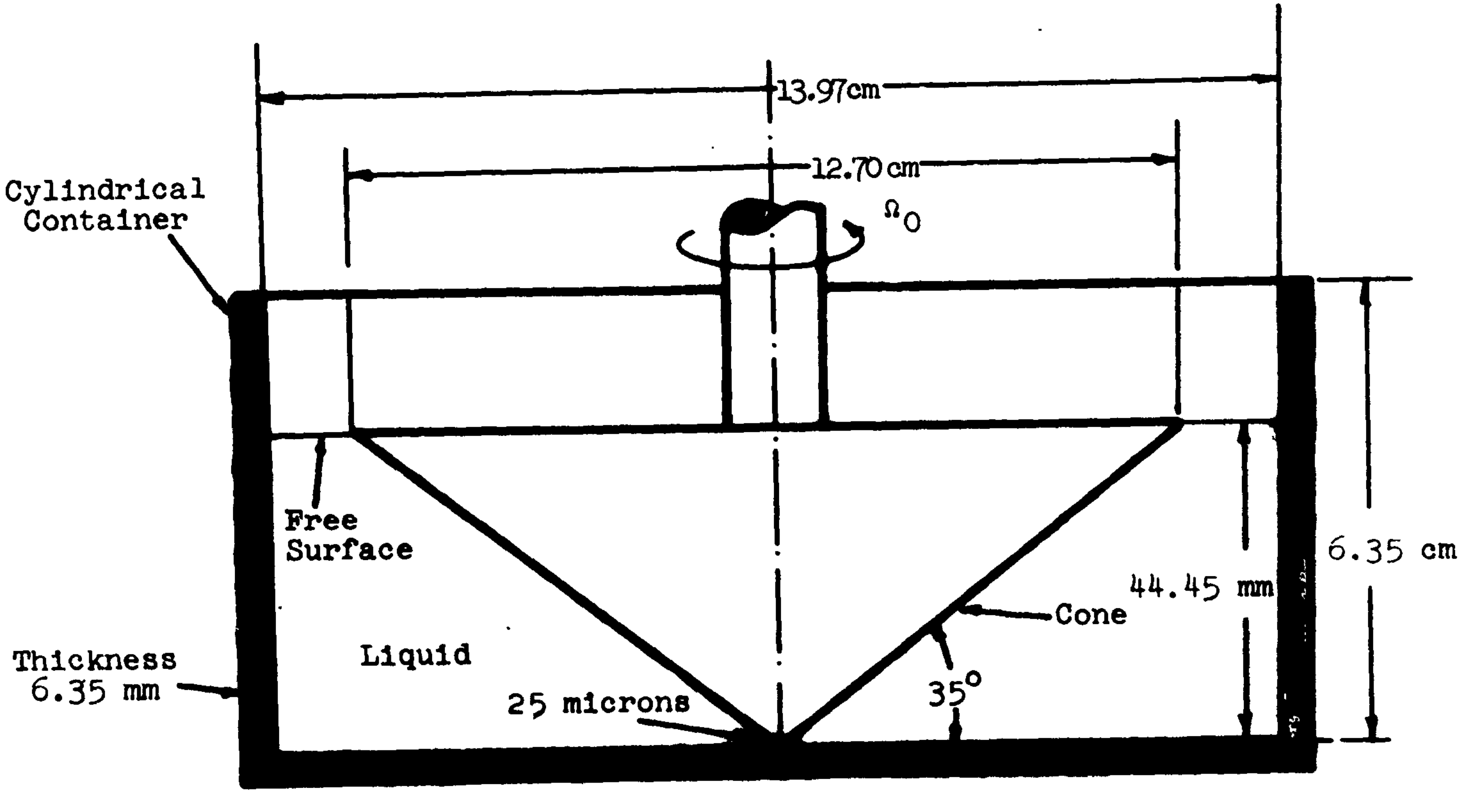

Figure 2. Dimensional Cross-Section of Flow Chamber with a 35 Degree Angle Cone

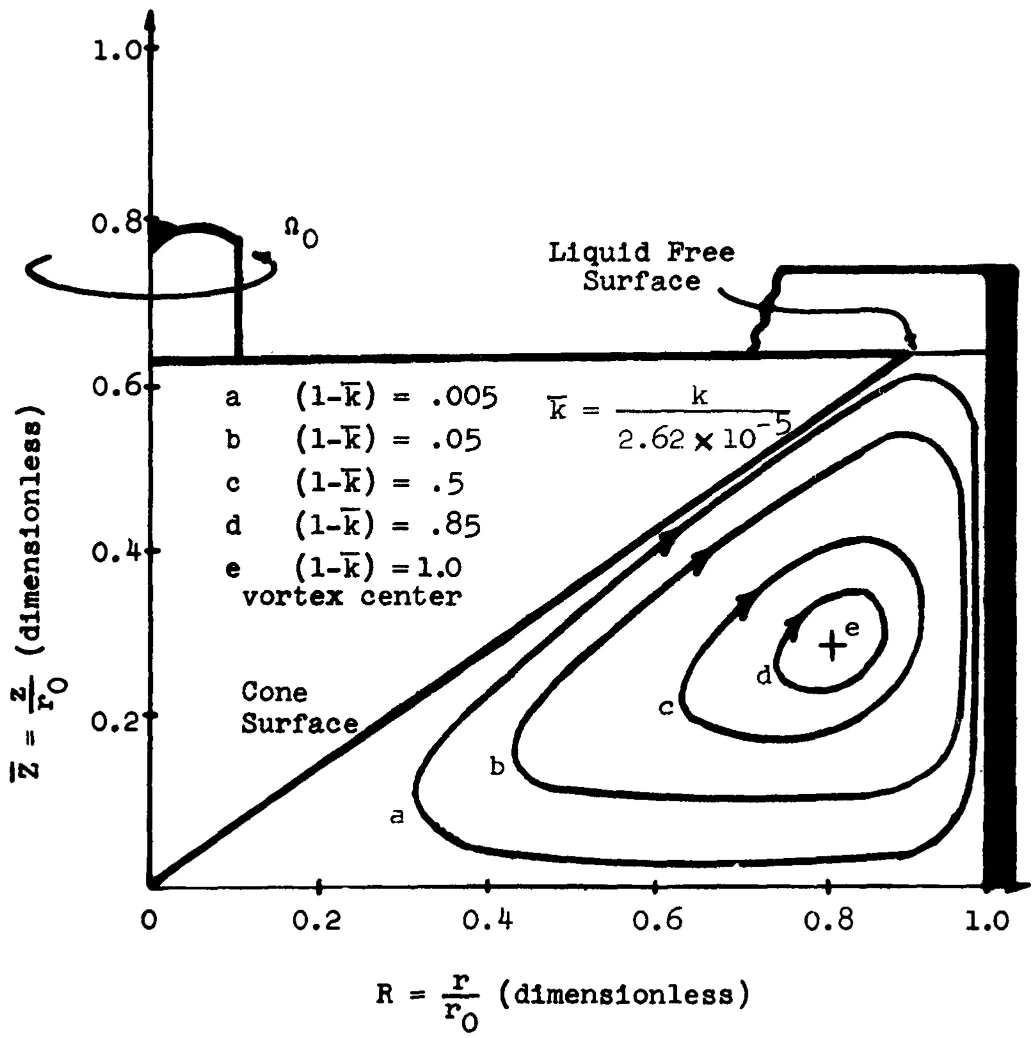

Figure 3. Velocity Trajectories in a ϕ = Constant Plane for $\alpha = 35^\circ$.

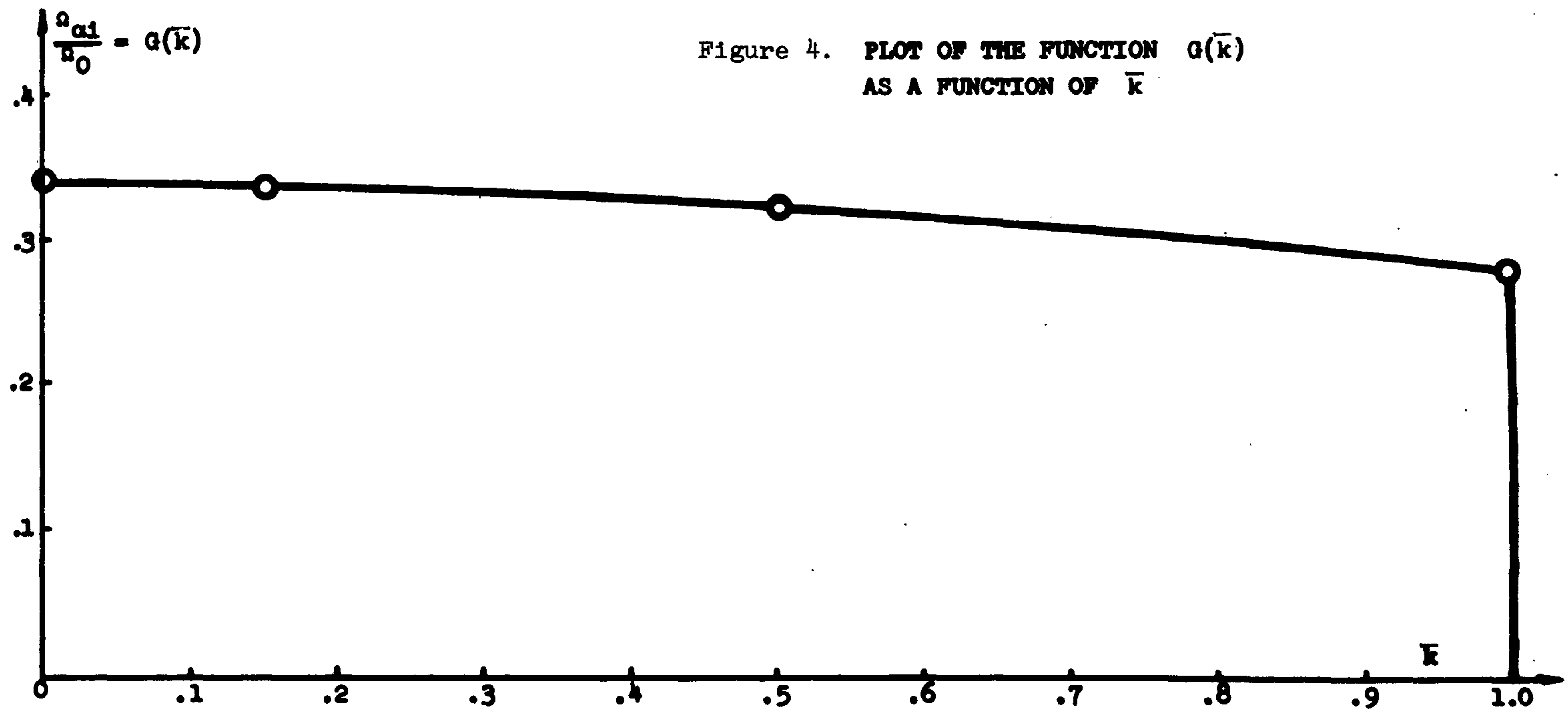

Figure 4. **PLOT OF THE FUNCTION $G(\bar{k})$ AS A FUNCTION OF $\bar{k}$**

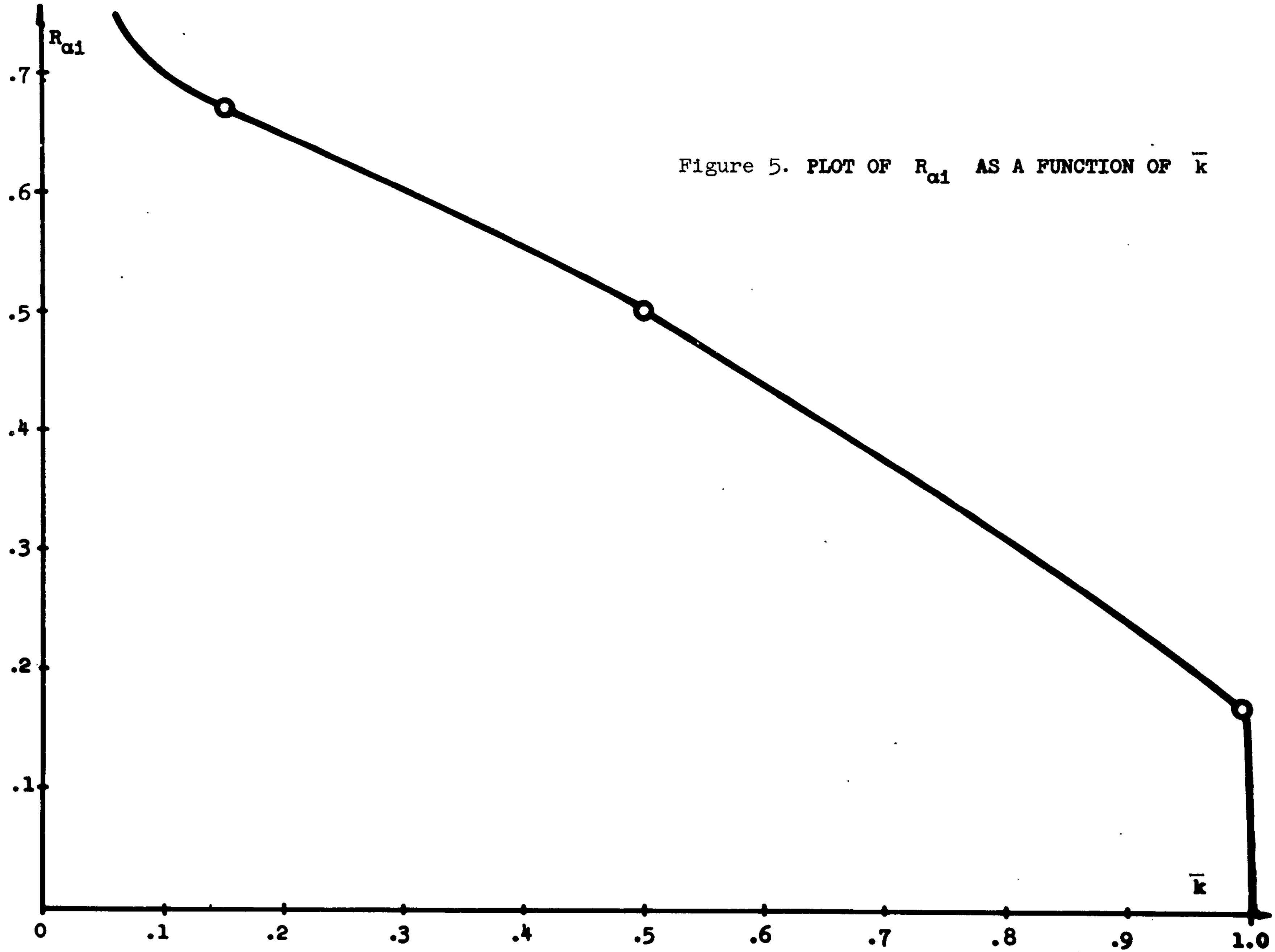

Figure 5. **PLOT OF $R_{\alpha 1}$ AS A FUNCTION OF $\bar{k}$**

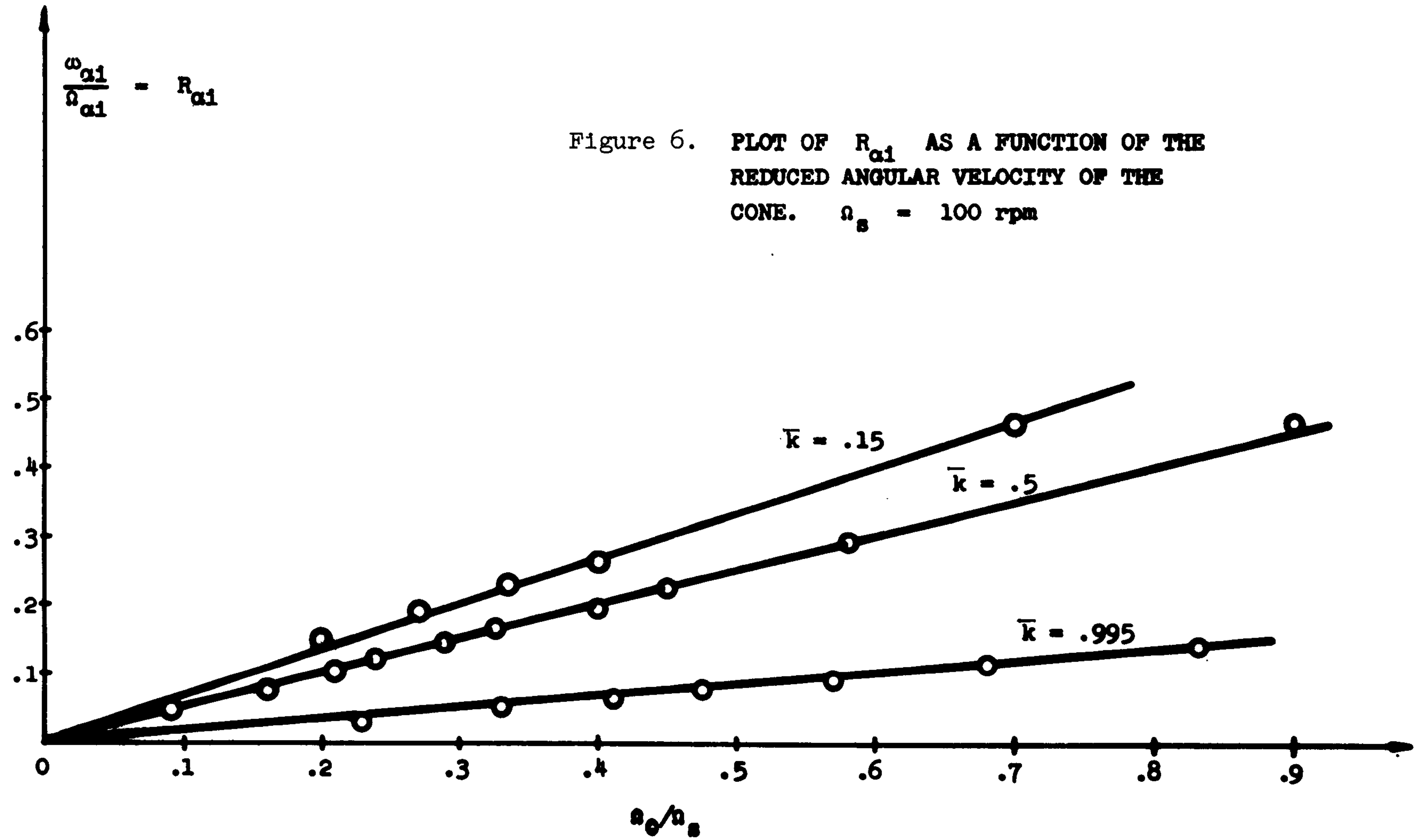

Figure 6. **PLOT OF $R_{\alpha i}$ AS A FUNCTION OF THE REDUCED ANGULAR VELOCITY OF THE CONE. Ω_s = 100 rpm**

α	ξ mm	η mm	r_c mm	z_c mm	R_c	Z_c	$\overline{Z}_c$	Ω'_c	$\Delta\Omega'_c$ %
45°	20.82	36.63	49.03	26.87	.772	.412	.423	.306	7.0
35°	16.74	44.12	53.11	19.38	.835	.436	.303	.348	3.5
10°	9.40	57.53	60.45	5.97	.952	.534	.094	.467	3.3
8°	9.14	58.93	60.71	4.57	.956	.509	.072	.499	2.0
6°	8.16	60.07	61.69	3.43	.968	.513	.054	.499	2.0
4°	6.86	61.30	62.99	2.20	.992	.497	.035	.500	1.6

Measurement of Coordinates of the Vortex Center and of Angular Velocity of Particles at the Vortex Center.

Table I

Material	$\alpha = 1^\circ$	$\alpha = 2^\circ$	$\alpha = 4^\circ$	$\alpha = 6^\circ$	$\alpha = 8^\circ$	$\alpha = 10^\circ$	$\alpha = 35^\circ$
Castor Oil	.329	.370	.392	.427	.447	.447	1.195
SAE 30	.334	.372	.391	.392	.412	.416	1.105
SAE 20-20W	.329	.349	.376	.403	.420	.442	1.139
SAE 10W	.329	.354	.368	.392	.410	.457	
Average Value	.331	.361	.382	.401	.418	.439	1.146

Values of $\dfrac{M/\Omega_0 \cdot \cos\alpha \sin\alpha}{2\pi\mu r_0^3}$ for Different Angled Cones.

Table II

α	1°	2°	4°	6°	8°	10°	35°
δ	-22.2	-21.2	-20.5	-19.9	-19.3	-18.6	+ 5.00

Values of δ for Different Angled Cones.

Table III

$T^{\circ}C$	Material	μ(cp)
22.2°	Castor Oil	870
22.2°	SAE 30	212
22.2°	SAE 20-20W	137
22.2°	SAE 10W	65.1

Calculated Viscosities

Table IV

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